

COMMON FIXED POINT THEOREMS FOR WEAKLY (ψ, S, C)-CONTRACTIVE MAPPINGS IN PARTIALLY ORDERED B -METRIC SPACES

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Abstract. *The purpose of this paper is to introduce the notion of weakly (ψ, S, C) -contractive mappings and to state some common fixed point theorems for these classes of mappings. These results are generalizations of the main results in [H. K. Nashine, Common fixed points via weakly (ψ, S, C) -contraction mappings on ordered metric spaces and application to integral equations, *Thai J. Math.* **12** (2014), no. 3, 729 – 747]. Also, some examples are given to illustrate the results.*

1 Introduction and preliminaries

In 1972, Chatterjea [7] introduced the notion of a C -contraction. This notion was generalized to a weak C -contraction by Choudhury [8] and a (μ, ψ) -generalized f -weakly contractive mapping in metric spaces by Chandok [5]. There were some fixed point results for (μ, ψ) -generalized f -weakly contractive mappings in complete metric spaces [5, Theorem 2.1], and in complete partially ordered metric space [6, Theorem 2.1]. In 2013, Dung and Hang [10] introduced the notion of a weak C -contraction mapping in partially ordered 2-metric spaces and stated some fixed point results for this mapping in complete partially ordered 2-metric spaces [10, Theorem 2.3, Theorem 2.4, Theorem 2.5]. In 2014, Nashine [14] generalized the notion of a weak C -contractive in metric spaces to a weakly (ψ, S, C) -contractive mappings and stated some common fixed point results for these classes of mappings.

There were many generalizations of a metric space and many fixed point theorems on generalized metric spaces were stated [2]. The notion of a b -metric space was introduced by Bakhtin [4] and then extensively used by Czerwik [9] as follows.

Definition 1.1 ([9]). Let X be a non-empty set and $d : X \times X \rightarrow [0, \infty)$ be a function such that for some $s \geq 1$ and all $x, y, z \in X$,

- (1) $d(x, y) = 0$ if and only if $x = y$.
- (2) $d(x, y) = d(y, x)$.
- (3) $d(x, y) \leq s(d(x, z) + d(z, y))$.

Then, d is called a b -metric on X and (X, d, s) is called a b -metric space.

Remark 1.2. (X, d) is a metric space if and only if $(X, d, 1)$ is a b -metric space.

The first important difference between a metric and a b -metric is that the b -metric need not be a continuous function in its two variables, see [13, Example 13]. In recent years, many fixed point theorems on b -metric spaces were stated, the readers may refer to [3, 11, 12, 16] and references therein.

The purpose of this paper is to introduce the notion of a weakly (ψ, S, C) -contractive mapping in partially ordered b -metric spaces and to state some common fixed point theorems for these classes of mappings. Also, some examples are given to illustrate the results.

First, we recall some notions and lemmas which will be useful in what follows.

Definition 1.3 ([9]). Let (X, d, s) be a b -metric space.

- (1) A sequence $\{x_n\}$ is called *convergent* to x , written as $\lim_{n \rightarrow \infty} x_n = x$, if $\lim_{n \rightarrow \infty} d(x_n, x) = 0$.
- (2) A sequence $\{x_n\}$ is called *Cauchy* in X if $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$.
- (3) (X, d, s) is called *complete* if every Cauchy sequence is a convergent sequence.

In 2014, Aghajani *et al.* [1] proved the following simple lemma about the convergence in b -metric spaces.

Lemma 1.4 ([1]). Let (X, d, s) be a b -metric space and $\lim_{n \rightarrow \infty} x_n = x, \lim_{n \rightarrow \infty} y_n = y$. Then

- (1) $\frac{1}{s^2}d(x, y) \leq \liminf_{n \rightarrow \infty} d(x_n, y_n) \leq \limsup_{n \rightarrow \infty} d(x_n, y_n) \leq s^2d(x, y)$. In particular, if $x = y$, then $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.
- (2) For each $z \in X$, $\frac{1}{s}d(x, z) \leq \liminf_{n \rightarrow \infty} d(x_n, z) \leq \limsup_{n \rightarrow \infty} d(x_n, z) \leq sd(x, z)$.

The following lemma is a equivalent condition for the Cauchy property of $\{x_n\}$ in b -metric spaces.

Lemma 1.5. Let (X, d, s) be a b -metric space and $\{x_n\}$ be a sequence in (X, d, s) . Then, the following statements are equivalent.

- (1) $\{x_n\}$ is a Cauchy sequence in (X, d, s) .
- (2) $\{x_{2n}\}$ is a Cauchy sequence in (X, d, s) and $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$.

Definition 1.6 ([17]). Let (X, \preceq) be a partially ordered set and $T, S : X \rightarrow X$ be two mappings.

- (1) The pair (T, S) is called *weakly increasing* if $Sx \preceq TSx$ and $Tx \preceq STx$ for all $x \in X$.
- (2) The mapping S is called *T -weakly isotone increasing* if $Sx \preceq TSx \preceq STSx$ for all $x \in X$.

Remark 1.7 ([14]). If the pair (T, S) is weakly increasing, then S is T -weakly isotone increasing.

Definition 1.8. Let (X, d, s, \preceq) be a partially ordered b -metric space. Then, (X, d, s, \preceq) is called a *regular space* if $\{z_n\}$ is a non-decreasing sequence in X and $\lim_{n \rightarrow \infty} z_n = z$, then $z_n \preceq z$ for all $n \in \mathbb{N} \cup \{0\}$.

2 Main results

First, we introduce the notion of a weakly (ψ, S, C) -contractive in partially ordered b -metric spaces. Denote by

- (1) Ψ the family of all increasing functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that $\psi(t) \leq \frac{1}{2}t$ for all $t \geq 0$. Notice that $\psi(0) = 0$.
- (2) Φ the family of all lower semi-continuous functions $\varphi : [0, \infty)^2 \rightarrow [0, \infty)$ such that $\varphi(x, y) = 0$ if and only if $x = y = 0$, and $\varphi(x, y) \leq x + y$ for all $x, y \in [0, \infty)$.

Definition 2.1. Let (X, d, s, \preceq) be a partially ordered b -metric space and $T, S : X \rightarrow X$ be two mappings. Then, T is called a *weakly (ψ, S, C) -contraction* if there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that for all $x, y \in X$ with $x \preceq y$ or $y \preceq x$,

$$d(Tx, Sy) \leq \psi\left(\frac{2}{s(s^2 + 1)}[d(x, Sy) + d(y, Tx) - \varphi(d(x, Sy), d(y, Tx))]\right). \quad (2.1)$$

Remark 2.2. A weakly (ψ, S, C) -contractive in [14] is a particular case of a weakly (ψ, S, C) -contractive in Definition 2.1 for $s = 1$.

The following lemma states the relation between the fixed point of T, S and the common fixed point of T, S . The proof of this lemma is immediate.

Lemma 2.3. *Let (X, d, s, \preceq) be a partially ordered b -metric space and $T, S : X \rightarrow X$ be two mappings satisfying the condition (2.1). If z is a fixed point of T or S , then z is a common fixed point of T and S .*

The following theorem is a sufficient condition for the existence and uniqueness of the common fixed point for a weakly (ψ, S, C) -contractive in b -metric spaces.

Theorem 2.4. *Let (X, d, s, \preceq) be a complete, partially ordered b -metric space and $T, S : X \rightarrow X$ be two mappings such that*

- (1) T is a weakly (ψ, S, C) -contraction.
- (2) S is T -weakly isotone increasing.
- (3) There exists x_0 such that $x_0 \preceq Sx_0$.
- (4) T or S is continuous, or (X, d, s, \preceq) is a regular space.

Then, T and S have a common fixed point. Moreover, the set of common fixed points of T, S is totally ordered if and only if T and S have a unique common fixed point.

Proof. Define a sequence $\{x_n\}$ in X by

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}$$

for all $n \in \mathbb{N} \cup \{0\}$, where x_0 be defined by the assumption (3). Since S is T -weakly isotone increasing, we have

$$x_0 \preceq x_1 \preceq \dots \preceq x_n \preceq x_{n+1} \preceq \dots$$

Sine $x_{2n} \preceq x_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, from (2.1), we have

$$\begin{aligned}
& d(x_{2n+1}, x_{2n+2}) \\
&= d(Tx_{2n+1}, Sx_{2n}) \\
&\leq \psi\left(\frac{2}{s(s^2+1)} [d(x_{2n+1}, Sx_{2n}) + d(x_{2n}, Tx_{2n+1}) \right. \\
&\quad \left. - \varphi(d(x_{2n+1}, Sx_{2n}), d(x_{2n}, Tx_{2n+1}))]\right) \\
&= \psi\left(\frac{2}{s(s^2+1)} [d(x_{2n+1}, x_{2n+1}) + d(x_{2n}, x_{2n+2}) \right. \\
&\quad \left. - \varphi(d(x_{2n+1}, x_{2n+1}), d(x_{2n}, x_{2n+2}))]\right) \\
&\leq \frac{1}{s(s^2+1)} [d(x_{2n}, x_{2n+2}) - \varphi(0, d(x_{2n}, x_{2n+2}))] \\
&\leq \frac{1}{s(s^2+1)} d(x_{2n}, x_{2n+2}) \\
&\leq \frac{1}{s^2+1} [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \\
&\leq \frac{1}{2} [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]. \tag{2.2}
\end{aligned}$$

It implies that

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) \tag{2.3}$$

for all $n \in \mathbb{N} \cup \{0\}$. Similarly, we also have

$$d(x_{2n+2}, x_{2n+3}) \leq d(x_{2n+1}, x_{2n+2}) \tag{2.4}$$

for all $n \in \mathbb{N} \cup \{0\}$. Therefore, from (2.3) and (2.4), we have $d(x_{n+1}, x_{n+1}) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of non-negative real numbers. Then, there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = r. \tag{2.5}$$

Taking the limit as $n \rightarrow \infty$ in (2.2) and using (2.5), we get

$$r \leq \lim_{n \rightarrow \infty} \frac{d(x_{2n}, x_{2n+2})}{s(s^2+1)} \leq \frac{r+r}{2} \leq r.$$

It implies that

$$\lim_{n \rightarrow \infty} d(x_{2n}, x_{2n+2}) = rs(s^2+1). \tag{2.6}$$

Taking the limit as $n \rightarrow \infty$ in (2.2), using (2.5), (2.6) and the lower semi-continuous property of φ , we have

$$r \leq \frac{rs(s^2+1) - \varphi(0, rs(s^2+1))}{s(s^2+1)} = r - \frac{\varphi(0, rs(s^2+1))}{s(s^2+1)} \leq r.$$

It implies that $\varphi(rs(s^2+1), 0) = 0$ and hence $r = 0$. Then (2.5) becomes

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. From Lemma 1.5 and (2.7), it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. Suppose to the contrary that $\{x_{2n}\}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{2n(k)}\}, \{x_{2m(k)}\}$ of $\{x_{2n}\}$ where $m(k)$ is a smallest integer such that $m(k) > n(k) > k$ and

$$d(x_{2n(k)}, x_{2m(k)}) \geq \varepsilon. \quad (2.8)$$

It implies that

$$d(x_{2n(k)}, x_{2m(k)-2}) < \varepsilon. \quad (2.9)$$

Then, from (2.8), we have

$$\begin{aligned} \varepsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\ &\leq sd(x_{2m(k)}, x_{2m(k)-2}) + sd(x_{2m(k)-2}, x_{2n(k)}) \\ &\leq sd(x_{2m(k)}, x_{2m(k)-2}) + s^2d(x_{2m(k)-2}, x_{2n(k)-1}) + s^2d(x_{2n(k)-1}, x_{2n(k)}). \end{aligned} \quad (2.10)$$

Taking the upper limit as $k \rightarrow \infty$ in (2.10) and using (2.7), we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{k \rightarrow \infty} d(x_{2n(k)-1}, x_{2m(k)-2}). \quad (2.11)$$

From (2.9), we have

$$\begin{aligned} d(x_{2n(k)-1}, x_{2m(k)-2}) &\leq sd(x_{2n(k)-1}, x_{2n(k)}) + sd(x_{2n(k)}, x_{2m(k)-2}) \\ &< sd(x_{2n(k)-1}, x_{2n(k)}) + \varepsilon s. \end{aligned} \quad (2.12)$$

Taking the upper limit as $k \rightarrow \infty$ in (2.12) and using (2.7), we get

$$\limsup_{k \rightarrow \infty} d(x_{2n(k)-1}, x_{2m(k)-2}) \leq \varepsilon s. \quad (2.13)$$

Therefore, from (2.11) and (2.13), we obtain

$$\frac{\varepsilon}{s^2} \leq \limsup_{k \rightarrow \infty} d(x_{2n(k)-1}, x_{2m(k)-2}) \leq \varepsilon s. \quad (2.14)$$

Similarly, we also have

$$\frac{\varepsilon}{s^2} \leq \liminf_{n \rightarrow \infty} d(x_{2n(k)-1}, x_{2m(k)-2}) \leq \varepsilon s. \quad (2.15)$$

Also, we have

$$d(x_{2n(k)-1}, x_{2m(k)-2}) \leq sd(x_{2n(k)-1}, x_{2n(k)}) + sd(x_{2n(k)}, x_{2m(k)-2}). \quad (2.16)$$

Taking the upper limit as $k \rightarrow \infty$ in (2.16) and using (2.7), (2.9), (2.14), we get

$$\frac{\varepsilon}{s^3} \leq \limsup_{k \rightarrow \infty} d(x_{2n(k)}, x_{2m(k)-2}) \leq \varepsilon. \quad (2.17)$$

Similarly, we also have

$$\frac{\varepsilon}{s^3} \leq \liminf_{k \rightarrow \infty} d(x_{2n(k)}, x_{2m(k)-2}) \leq \varepsilon. \quad (2.18)$$

Again, we have

$$\begin{aligned}
& d(x_{2m(k)-1}, x_{2n(k)-1}) \\
& \leq sd(x_{2m(k)-1}, x_{2n(k)}) + sd(x_{2n(k)}, x_{2n(k)-1}) \\
& \leq s^2 d(x_{2m(k)-1}, x_{2m(k)-2}) + s^2 d(x_{2m(k)-2}, x_{2n(k)}) + sd(x_{2n(k)}, x_{2n(k)-1}) \\
& < s^2 d(x_{2m(k)-1}, x_{2m(k)-2}) + s^2 \varepsilon + sd(x_{2n(k)}, x_{2n(k)-1}).
\end{aligned} \tag{2.19}$$

Taking the upper limit as $k \rightarrow \infty$ in (2.19) and using (2.7), we get

$$\limsup_{k \rightarrow \infty} d(x_{2n(k)-1}, x_{2m(k)-1}) \leq \varepsilon s^2. \tag{2.20}$$

Also, we have

$$d(x_{2n(k)-1}, x_{2m(k)-2}) \leq sd(x_{2n(k)-1}, x_{2m(k)-1}) + sd(x_{2m(k)-1}, x_{2m(k)-2}). \tag{2.21}$$

Taking the upper limit as $k \rightarrow \infty$ in (2.21) and using (2.7), (2.11), we get

$$\frac{\varepsilon}{s^3} \leq \limsup_{k \rightarrow \infty} d(x_{2n(k)-1}, x_{2m(k)-1}). \tag{2.22}$$

Therefore, from (2.20) and (2.22), we have

$$\frac{\varepsilon}{s^3} \leq \limsup_{k \rightarrow \infty} d(x_{2n(k)-1}, x_{2m(k)-1}) \leq s^2 \varepsilon. \tag{2.23}$$

Similarly, we also have

$$\frac{\varepsilon}{s^3} \leq \liminf_{k \rightarrow \infty} d(x_{2n(k)-1}, x_{2m(k)-1}) \leq s^2 \varepsilon. \tag{2.24}$$

Since $x_{2m(k)-2} \succeq x_{2n(k)-1}$, from (2.1), we have

$$\begin{aligned}
& d(x_{2n(k)}, x_{2m(k)}) \\
& \leq sd(x_{2n(k)}, x_{2m(k)-1}) + sd(x_{2m(k)-1}, x_{2m(k)}) \\
& = sd(Tx_{2n(k)-1}, Sx_{2m(k)-2}) + sd(x_{2m(k)-1}, x_{2m(k)}) \\
& \leq sd(x_{2m(k)-1}, x_{2m(k)}) + s\psi \left(\frac{2}{s(s^2+1)} [d(x_{2n(k)-1}, Sx_{2m(k)-2}) \right. \\
& \quad \left. + d(x_{2m(k)-2}, Tx_{2n(k)-1}) - \varphi(d(x_{2n(k)-1}, Sx_{2m(k)-2}), \right. \\
& \quad \left. d(x_{2m(k)-2}, Tx_{2n(k)-1})) \right] \Big) \\
& \leq sd(x_{2m(k)-1}, x_{2m(k)}) + s\psi \left(\frac{2}{s(s^2+1)} [d(x_{2n(k)-1}, x_{2m(k)-1}) \right. \\
& \quad \left. + d(x_{2m(k)-2}, x_{2n(k)}) - \varphi(d(x_{2n(k)-1}, x_{2m(k)-1}), d(x_{2m(k)-2}, x_{2n(k)}))] \right) \\
& \leq sd(x_{2m(k)-1}, x_{2m(k)}) + \frac{1}{s^2+1} [d(x_{2n(k)-1}, x_{2m(k)-1}) + d(x_{2m(k)-2}, x_{2n(k)}) \\
& \quad - \varphi(d(x_{2n(k)-1}, x_{2m(k)-1}), d(x_{2m(k)-2}, x_{2n(k)}))] .
\end{aligned} \tag{2.25}$$

Taking the upper limit as $k \rightarrow \infty$ in (2.25) and using (2.7), (2.8), (2.17), (2.18), (2.23), (2.24) and the lower semi-continuous property of φ , we get

$$\begin{aligned} \varepsilon &\leq \frac{1}{s^2 + 1} \left[\varepsilon s^2 + \varepsilon - \liminf_{k \rightarrow \infty} \varphi(d(x_{2n(k)-1}, x_{2m(k)-1}), d(x_{2n(k)}, x_{2m(k)-2})) \right] \\ &\leq \varepsilon - \frac{1}{s^2 + 1} \varphi \left[\liminf_{k \rightarrow \infty} d(x_{2n(k)-1}, x_{2m(k)-1}), \liminf_{k \rightarrow \infty} d(x_{2n(k)}, x_{2m(k)-2}) \right] \\ &< \varepsilon. \end{aligned}$$

It is a contradiction. Thus, $\{x_{2n}\}$ is a Cauchy sequence. By Lemma 1.5, $\{x_n\}$ is a Cauchy sequence in (X, d, s) . Since (X, d, s) is complete, there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$.

Suppose that T or S is continuous. If T is continuous, then

$$z = \lim_{n \rightarrow \infty} x_{2n+2} = \lim_{n \rightarrow \infty} Tx_{2n+1} = T(\lim_{n \rightarrow \infty} x_{2n+1}) = Tz,$$

that is, z is a fixed point of T . By Lemma 2.3, z is a common fixed point of S and T . Similarly, if S is continuous, we also see that z is a common fixed point of S and T .

Suppose that (X, d, s, \preceq) is a regular space. Then $x_{2n+1} \preceq z$ for all $n \geq 0$. From (2.1), we have

$$\begin{aligned} &d(x_{2n+2}, Sz) \\ &= d(Tx_{2n+1}, Sz) \\ &\leq \psi \left(\frac{2}{s(s^2 + 1)} [d(x_{2n+1}, Sz) + d(z, Tx_{2n+1}) - \varphi(d(x_{2n+1}, Sz), d(z, Tx_{2n+1}))] \right) \\ &\leq \psi \left(\frac{2}{s(s^2 + 1)} [d(x_{2n+1}, Sz) + d(z, x_{2n+2}) - \varphi(d(x_{2n+1}, Sz), d(z, x_{2n+2}))] \right) \\ &\leq \frac{1}{s(s^2 + 1)} [d(x_{2n+1}, Sz) + d(z, x_{2n+2}) - \varphi(d(x_{2n+1}, Sz), d(z, x_{2n+2}))] \\ &\leq \frac{1}{s(s^2 + 1)} [d(x_{2n+1}, Sz) + d(z, x_{2n+2})]. \end{aligned} \tag{2.26}$$

Taking the upper limit as $n \rightarrow \infty$ in (2.26), using $\lim_{n \rightarrow \infty} x_n = z$ and Lemma 1.4, we get

$$\frac{1}{s} d(z, Sz) \leq \frac{1}{s(s^2 + 1)} [sd(z, Sz)] = \frac{1}{s^2 + 1} d(z, Sz).$$

It implies that

$$d(z, Sz) \leq \frac{s}{s^2 + 1} d(z, Sz) \leq \frac{1}{2} d(z, Sz).$$

This implies that $d(z, Sz) = 0$ and hence $Sz = z$, that is, z is a fixed point of S . Therefore, from Lemma 2.3, it follows that z is a common fixed point of T and S .

Now, suppose that the set of common fixed points of T and S is totally ordered. We claim that there is a unique common fixed point of T and S . If there exist $u, v \in X$

such that $Su = Tu = u$ and $Sv = Tv = v$, then, from (2.1), we have

$$\begin{aligned}
& d(u, v) = d(Tu, Sv) \\
& \leq \psi\left(\frac{2}{s(s^2 + 1)} [d(u, Sv) + d(v, Tu) - \varphi(d(u, Sv), d(v, Tu))]\right) \\
& = \psi\left(\frac{2}{s(s^2 + 1)} [(2d(u, v) - \varphi(d(u, v), d(v, u)))]\right) \\
& \leq \frac{1}{s(s^2 + 1)} [2d(u, v) - \varphi(d(u, v), d(v, u))] \\
& = \frac{2}{s(s^2 + 1)} d(u, v) - \frac{1}{s(s^2 + 1)} \varphi(d(u, v), d(v, u)) \\
& \leq d(u, v) - \frac{1}{s(s^2 + 1)} \varphi(d(u, v), d(v, u)).
\end{aligned}$$

It implies that $\varphi(d(u, v), d(v, u)) = 0$ and hence $d(u, v) = 0$. Therefore, $u = v$, that is, the common fixed point of T and S is unique. Conversely, if T and S have a unique common fixed point, then the set of common fixed points of T and S being a singleton is totally ordered. \square

By using Remark 1.7, from Theorem 2.4, we get the following corollary.

Corollary 2.5. *Let (X, d, s, \preceq) be a complete, partially ordered b-metric space and $T, S : X \rightarrow X$ be two mappings such that*

- (1) *T is a weakly (ψ, S, C) -contraction.*
- (2) *The pair (T, S) is weakly increasing.*
- (3) *There exists x_0 such that $x_0 \preceq Sx_0$.*
- (4) *S or T is continuous, or (X, d, s, \preceq) is a regular space.*

Then, T and S have a common fixed point. Moreover, the set of fixed points of T, S is totally ordered if and only if T and S have a unique common fixed point.

By taking $\psi = \frac{x}{2}$ and $\varphi(x, y) = (1 - \alpha)(x + y)$ for all $x, y \in [0, \infty)$ and for some $\alpha \in [0, 1)$ in Corollary 2.5, we get the following corollary.

Corollary 2.6. *Let (X, d, s, \preceq) be a complete, partially ordered b-metric space and $T, S : X \rightarrow X$ be two mappings such that*

- (1) *There exists $\lambda \in [0, \frac{1}{s(s^2+1)})$ such that for all $x, y \in X$ with $x \preceq y$ or $x \succeq y$,*

$$d(Tx, Sy) \leq \lambda(d(x, Sy) + d(y, Tx)).$$

- (2) *The pair (T, S) is weakly increasing.*
- (3) *There exists x_0 such that $x_0 \preceq Sx_0$.*
- (4) *T or S is continuous, or (X, d, s, \preceq) is a regular space.*

Then, T and S have a common fixed point. Moreover, the set of fixed points of T, S is totally ordered if and only if T and S have a unique common fixed point.

In the next, we give some examples to support our results. The following example is an illustration of Theorem 2.4.

Example 2.7. Let $X = \{1, 2, 3, 4, 5\}$ with the usual order \leq . Define a function d on X as follows.

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } (x, y) \in \{(1, 2), (2, 1)\} \\ 20 & \text{if } (x, y) \in \{(3, 2), (2, 3), (4, 2), (2, 4), (5, 2), (2, 5)\} \\ 9 & \text{otherwise.} \end{cases}$$

Then, (X, d, s) is a complete b -metric space with $s = 2$. Let $T, S : X \rightarrow X$ be defined by $T1 = T2 = T3 = T4 = T5 = 2$ and $S1 = S2 = 2, S3 = S4 = S5 = 1$. Define two functions by $\varphi(x, y) = \frac{x + y}{2}$ and $\psi(x) = \frac{x}{2}$ for all $x, y \in [0, \infty)$. Then, $\varphi \in \Phi, \psi \in \Psi$. Put

$$\begin{aligned} P &= \psi\left(\frac{2}{s(s^2 + 1)}[d(x, Sy) + d(y, Tx) - \varphi(d(x, Sy), d(y, Tx))]\right) \\ &= \psi\left(\frac{1}{5}[d(x, Sy) + d(y, Tx) - \varphi(d(x, Sy), d(y, Tx))]\right) \\ &= \frac{1}{20}[d(x, Sy) + d(y, Tx)]. \end{aligned}$$

Let $x, y \in X$ with $x \leq y$ or $y \geq x$, we have

$$d(Tx, Sy) = \begin{cases} 0 & \text{if } x, y \in \{1, 2\} \text{ or } x \in \{3, 4, 5\}, y \in \{1, 2\} \\ 1 & \text{if } x \in \{1, 2\}, y \in \{3, 4, 5\} \text{ or } x, y \in \{3, 4, 5\} \end{cases}$$

and

$$P = \begin{cases} 0 & \text{if } x = y = 2 \\ 1 & \text{if } x = 1, y \in \{3, 4, 5\} \text{ or } x \in \{3, 4, 5\}, y = 2 \\ \frac{1}{10} & \text{if } x = y = 1 \\ \frac{1}{20} & \text{if } (x, y) \in \{(1, 2), (2, 1)\} \\ \frac{21}{20} & \text{if } x = 2, y \in \{3, 4, 5\} \text{ or } x \in \{3, 4, 5\}, y = 1 \\ \frac{29}{20} & \text{if } x, y \in \{3, 4, 5\}. \end{cases}$$

It implies that the condition (2.1) holds and hence the assumption (1) of Theorem 2.4 is satisfied. Moreover, other assumptions of Theorem 2.4 are fulfilled. Therefore, Theorem 2.4 is applicable to T, S, φ, ψ and (X, d, s, \leq) .

However, since $20 = d(2, 3) \geq d(2, 1) + d(1, 3) = 10$, d is not a metric on X . Thus, [14, Theorem 3.2] is not applicable to (X, d, s) .

Finally, we apply Corollary 2.6 to study the existence of solutions to the system of nonlinear integral equations.

Example 2.8. Let $C[a, b]$ be the set of all continuous function on $[a, b]$, the b -metric d with $s = 2^{p-1}$ be defined by

$$d(u, v) = \sup_{t \in [a, b]} \{|u(t) - v(t)|^p\}$$

for all $u, v \in C[a, b]$ and some $p > 1$, and the partial order \preceq be given by $u \preceq v$ if $u(t) \leq v(t)$ for all $t \in [a, b]$. Consider the system of nonlinear integral equations

$$\begin{cases} u(t) = \int_a^b K_1(t, s, u(s))ds + g(t) \\ u(t) = \int_a^b K_2(t, s, u(s))ds + g(t) \end{cases} \quad (2.27)$$

where $t \in [a, b], g : [a, b] \rightarrow \mathbb{R}, K_1, K_2 : [a, b] \times [a, b] \times u[a, b] \rightarrow \mathbb{R}$ for each $u \in C[a, b]$. Suppose that the following statements hold.

- (1) $K_1(t, s, u(s))$ and $K_2(t, s, u(s))$ are integrable with respect to s on $[a, b]$.
- (2) $Tu, Su \in C[a, b]$ for all $u \in C[a, b]$, where

$$Tu(t) = \int_a^b K_1(t, s, u(s))ds + g(t),$$

$$Su(t) = \int_a^b K_2(t, s, u(s))ds + g(t)$$

for all $t \in [a, b]$.

- (3) For all $t, s \in [a, b], u \in C[a, b]$,

$$K_1(t, s, u(t)) \leq K_2\left(t, s, \int_a^b K_1(s, z, u(z))dz + g(s)\right),$$

$$K_2(t, s, u(t)) \leq K_1\left(t, s, \int_a^b K_2(s, z, u(z))dz + g(s)\right).$$

- (4) For all $s, t \in [a, b]$ and $u, v \in C[a, b]$ with $u \preceq v$ or $v \preceq u$,

$$|K_1(t, s, u(s)) - K_2(t, s, v(s))|^p \leq \alpha(t, s)(|u(s) - Sv(s)|^p + |v(s) - Tu(s)|^p)$$

where $\alpha : [a, b] \times [a, b] \rightarrow [0, \infty)$ is a continuous function satisfying

$$\sup_{t \in [a, b]} \left(\int_a^b \alpha(t, s)ds \right) \leq \frac{1}{2^{p-1}(2^{2p-2} + 1)(b-a)^{p-1}}.$$

- (5) There exists $u_0 \in C[a, b]$ such that $u_0(t) \leq \int_a^b K_2(t, s, u_0(s))ds + g(t)$ for all $t \in [a, b]$.

Then, the system of nonlinear integral equations (2.27) has a solution $u \in C[a, b]$.

Proof. Consider $T, S : C[a, b] \rightarrow C[a, b]$ defined by

$$Tu(t) = \int_a^b K_1(t, s, u(s))ds + g(t)$$

and

$$Su(t) = \int_a^b K_2(t, s, u(s))ds + g(t)$$

for all $u \in C[a, b]$ and $t \in [a, b]$. It follows from the assumptions (1) and (2) that T and S are well-defined. Notice that the existence of a solution to (2.27) is equivalent to the existence of the common fixed point of T and S . Now, we prove that all assumptions of Corollary 2.6 are satisfied.

(1). Let $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. For all $u, v \in C[a, b]$ with $v \preceq u$ or $u \preceq v$, from the assumption (4), we have

$$\begin{aligned} & |Tu(t) - Sv(t)|^p \\ & \leq \left(\int_a^b |K_1(t, s, u(s)) - K_2(t, s, v(s))| ds \right)^p \\ & \leq \left[\left(\int_a^b ds \right)^{\frac{1}{q}} \left(\int_a^b |K_1(t, s, u(s)) - K_2(t, s, v(s))|^p ds \right)^{\frac{1}{p}} \right]^p \\ & \leq (b-a)^{\frac{p}{q}} \left(\int_a^b \alpha(t, s) (|u(s) - Sv(s)|^p + |v(s) - Tu(s)|^p) ds \right) \\ & \leq (b-a)^{p-1} \left(\int_a^b \alpha(t, s) (d(u, Sv) + d(v, Tu)) ds \right) \\ & \leq (b-a)^{p-1} (d(u, Sv) + d(v, Tu)) \left(\int_a^b \alpha(t, s) ds \right) \\ & \leq \lambda (d(u, Sv) + d(v, Tu)) \\ & = d(u, Sv) + d(v, Tu) - (1 - \lambda) (d(u, Sv) + d(v, Tu)). \end{aligned}$$

where $\lambda = (b-a)^{p-1} \sup_{t \in [a, b]} \left(\int_a^b \alpha(t, s) ds \right)$. It implies that

$$0 \leq \lambda < \frac{1}{2^{p-1}(2^{2p-2} + 1)}$$

and

$$d(Tx, Sy) \leq (d(u, Sv) + d(v, Tu)) - (1 - \lambda) (d(u, Sv) + d(v, Tu)).$$

Therefore, the assumption (1) in Corollary 2.6 holds with

$$\lambda = (b-a)^{p-1} \sup_{t \in [a, b]} \left(\int_a^b \alpha(t, s) ds \right).$$

(2). For all $u \in C[a, b]$ and all $t \in [a, b]$, from the assumption (3), we have

$$\begin{aligned} Tu(t) &= \int_a^b K_1(t, s, u(s)) ds \\ &\leq \int_a^b K_2 \left(t, s, \int_a^b K_1(s, z, u(z)) dz + g(s) \right) ds \\ &\leq \int_a^b K_2(t, s, Tu(s)) ds \\ &= STu(t) \end{aligned}$$

and

$$\begin{aligned}
 Su(t) &= \int_a^b K_2(t, s, u(s)) ds \\
 &\leq \int_a^b K_1\left(t, s, \int_a^b K_1(s, z, u(z)) dz + g(s)\right) ds \\
 &\leq \int_a^b K_1(t, s, Su(s)) ds \\
 &= TSu(t).
 \end{aligned}$$

It implies that $Tu \preceq STu$ and $Su \preceq TSu$ for all $u \in C[a, b]$. Therefore, the pair (T, S) is weakly increasing.

(3). From the assumption (5), there exists $x_0 \in C[a, b]$ such that $x_0 \preceq Sx_0$.

(4). By using the similar argument as in the proof of [15, Theorem 3.1], we also see that the space (X, d, s, \preceq) is regular.

By the above, all assumptions of Corollary 2.6 are satisfied. Then, T and S have a common fixed point $u \in C[a, b]$ and the system of integral equations (2.27) has a solution $u \in C[a, b]$. \square

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